Quantum chaos in the atomic gravitational cavity

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We report quantum chaos phenomena in the atomic gravitational cavity. We consider the reflection of cold atoms from a temporally modulated evanescent wave. In the globally chaotic regime, for small modulation, the squared energy distribution as a function of time demonstrates dynamical localization. However, for larger modulation delocalization occurs. [S1063-651X(97)11607-5]

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How will a classically chaotic system behave quantum mechanically? This has been the topic for quantum chaos for tens of years. According to Casati *et al.* [1], when a classical system is subjected to strong time-periodic perturbation, its motion becomes globally chaotic, and it will indefinitely absorb energy from the external field in a diffusive way. However, this ''diffusive absorption'' is suppressed by the quantum interference effect. This is the so-called dynamical localization phenomenon, which is the analogy of Anderson localization of solid state physics. This effect has been discussed in model systems in quantum chaos such as the kicked rotator [2], quantum bouncer with infinite potential [3], and atomic [4] and molecular [5] models. There is also experimental evidence that it occurs in hydrogen atoms in Rydberg states driven by a strong microwave field [6,7] and atoms in time-dependent standing wave [8,9]. Theoretically, under appropriate conditions, it should appear in periodically driven quantum systems, which are chaotic in their classical limit. Proposals for observing the effect include periodically driven Josephson junctions [10] and optical fibres with periodically varying index of refraction [11].

Dynamical localization is a very general phenomenon in periodically driven systems, but it will not occur under conditions of resonance or in special cases distinguished by some form of translational invariance [12,13]. The ''delocalization'' in such cases is a purely quantum effect: the long-time unbounded propagation is not related to the corresponding classical diffusion [12]. But is there a quasiclassical unbounded diffusion in the quantum context? By means of studying the dynamics of a quantum particle in a triangular potential well under a monochromatic driving, Casati *et al.* [14,15] found that a simple, physically relevant model exhibits a transition from a localized regime to a delocalized one as the strength of the monochromatic perturbation is increased beyond a quantum border in addition to the classical chaotic one. In Ref. [16], the delocalization in systems such as this was explained in terms of a characteristic change in the structure of the Floquet eigenstates of the system. Such behavior is generic for systems driven by a high frequency field, in which the density of states is slowly varying. Precisely such conditions prevail in the atomic gravitational cavity.

The system discussed in [14,15] models the dynamics of a particle elastically bouncing on a fixed wall at $x=0$ under a constant field plus a monochromatic perturbation and is described by the following Hamiltonian:

$$H(x,p,t)=\frac{p^2}{2}+\varepsilon_0 x+\varepsilon x\cos\omega t,\quad x\geqslant 0, \tag{1}$$

where x and p are the position and momentum variables, ε_0 and ε are the constant field strength and the perturbation amplitude, and ω is the perturbation frequency.

The condition for the onset of global chaotic behavior leading to dynamical localization is $\varepsilon_l>\varepsilon_0/4$. And the threshold for the transition from localization to delocalization, that is the delocalization border, is $\varepsilon_{\mathrm{dl}}\cong 0.5\ \omega^{3/2}$ provided that the condition of classical chaos $\varepsilon_l>\varepsilon_0/4$ is also satisfied.

The system we discuss is the atomic gravitational cavity [17], consisting of atoms bouncing vertically from an evanescent light field, which provides a reflection potential step varying rapidly with the distance from the boundary. The amplitude of the light field is periodically modulated in time. Atoms released from a magneto-optical trap (MOT) form a cold ''beam.'' The cold atoms drop vertically, and are reflected by the evanescent wave. We only consider the dynamics in the direction normal to the evanescent field; in terms of dimensionless variables, without modulation, the Hamiltonian of the system can be written as

$$H=\frac{p^2}{2}+\lambda x+\kappa e^{-x} \tag{2}$$

with the canonical commutation relations $[x,p]=iK$ [18].

We are using dimensionless position and momentum variables defined by $x=\alpha z$, where z is the displacement from the dielectric surface supporting the evanescent wave and α is the decay rate of the evanescent wave; $p=\alpha p_z/m\omega_s$ (where ω_s is a frequency scaling parameter, m is the mass of the atom, and p_z is the vertical momentum component along the z axis). While the scaled gravitational acceleration is $\lambda=\alpha g/\omega_s^2$, $\kappa=E\alpha^2/m\omega_s^2$, where $E=\hbar|\Omega_r|^2/\Delta$ is the amplitude of the evanescent potential in terms of the Rabi frequency Ω_r and the detuning of the laser from the atomic transition Δ; and $K=\hbar\alpha^2/m\omega_s$ is the dimensionless Planck constant. The reference frequency ω_s is chosen to be the recoil energy related frequency, that is, $\omega_s=\hbar\alpha^2/mK$. Time modulation of the evanescent field can be included by making κ time dependent, that is, $\kappa(\tau)=\kappa(1+\varepsilon_m\cos\Omega\tau)$, where Ω is the scaled driving frequency, ε_m (typically less than 1) is the modulation strength, and $\tau=\omega_s t$ is the scaled

time variable. As the difference between our system and the one in [14,15] is just a canonical transformation, we are motivated to find similar phenomena in this important physical system.

If the modulation strength is very small, the Hamiltonian of our system becomes

$$H=\frac{p^2}{2}+\lambda x+\kappa e^{-x+\varepsilon_m \cos \Omega\tau}. \tag{3}$$

Adopting the following canonical transformation:

$$X=\frac{\partial F}{\partial P}=x-\varepsilon_m \cos \Omega\tau, \tag{4}$$

$$p=\frac{\partial F}{\partial x}=P+\varepsilon_m\Omega \sin \Omega\tau \tag{5}$$

we have the new Hamiltonian $H_n=P^2/2+X(\lambda-\varepsilon_m\Omega^2 \cos \Omega\tau)+\kappa e^{-X}$. Compare with Eq. (1); if we choose $\omega=\Omega$, we have the following correspondence:

$$\lambda\rightarrow\varepsilon_0 \quad \text{and} \quad \varepsilon_m\Omega^2\rightarrow\varepsilon. \tag{6}$$

As a result the thresholds for localization and delocalization in our system are

$$\varepsilon_m>\frac{\lambda}{4\Omega^2} \quad \text{and} \quad \varepsilon_m>\frac{1}{2\sqrt{\Omega}}. \tag{7}$$

We would like to point out that different systems may have different variables that show localization and delocalization. In the system under discussion, through trials and comparisons, we find that the energy squared is a good indicator of localization.

Classically the system is described by Hamilton's equations:

$$\dot{x}=p, \tag{8}$$

$$\dot{p}=-\lambda+\kappa(1+\varepsilon_m \cos \Omega\tau)e^{-x}, \tag{9}$$

where an overdot indicates differentiation with respect to the scaled time. We solve the above equations numerically by means of the fourth-order symplectic integrator [19]. This is a periodically driven nonlinear oscillator. Generally such systems exhibit regions of regular and chaotic motion in phase space, depending on the modulation strength. According to KAM theorem [20], if the perturbation is small enough, most of the phase-space curves of the unperturbed model will remain but become distorted; for moderate perturbation, the phase space becomes a mixture of stochastic and regular motion; for large perturbation, the motion becomes globally chaotic. A stroboscopic portrait or Poincaré section [21] is proven to be an effective technique to view the complicated motion in phase space. We need to work in the globally chaotic regime to see dynamical localization.

In order to calculate the mean of energy squared, we have to find a proper way to describe the atomic dynamics. We will use probability distributions in phase space. We define a classical state to be a probability measure on phase space in the form of $Q(x,p)dx\,dp$ where $Q(x,p)$ is the joint probability density. The density then obeys the Liouville equation

$$\frac{\partial Q}{\partial\tau}=-\{H,Q\}_{x,p}=p\frac{\partial Q}{\partial x}-[\lambda-\kappa(1+\varepsilon_m \cos \Omega\tau)e^{-x}]\frac{\partial Q}{\partial p}, \tag{10}$$

where $\{,\}_{x,p}$ is the usual Poisson bracket. This equation can be solved by the method of characteristics [22]. We choose the initial state $Q_0(x,p)$ to be the Wigner function, which is a bivariate Gaussian centered on (x_0,p_0) with position variance σ_x and momentum variance σ_p:

$$Q_0(x,p)=\frac{1}{2\pi\sqrt{\sigma_x\sigma_p}}\exp\left[\frac{(x-x_0)^2}{2\sigma_x}\right]\exp\left[\frac{(p-p_0)^2}{2\sigma_p}\right]. \tag{11}$$

The solution to Eq. (10) is

$$Q(x,p,\tau)=Q_0[\overline{x}(x,p,-\tau),\overline{p}(x,p,-\tau)], \tag{12}$$

where $[\overline{x}(x,p,-\tau),\overline{p}(x,p,-\tau)]$ is the trajectory generated by Hamilton's equations (8) and (9) .

In order to calculate the mean of energy squared as a function of time, we track 2000 points in phase space initially distributed with a density given by Eq. (11). Because we are just interested in long time evolution, we will plot the statistics at times $\tau=(2\pi/\Omega)s$, where s is an integer referred to as the strobe number.

Quantum mechanically the system is governed by the time dependent Schrödinger equation $iK(d/d\tau)|\psi\rangle=H|\psi\rangle$, where $H=p^2/2+\lambda x+\kappa(1+\varepsilon_m \cos \Omega\tau)e^{-x}$, and is solved numerically by the second order split-operator method [23]. The calculation of the mean of the energy squared is carried out in the following way:

The time independent part of the Hamiltonian is $H_0=p^2/2+V(x)$, where $V(x)=\lambda x+\kappa e^{-x}$. So the mean of the energy squared is given by

$$\langle H_0^2\rangle=\left\langle\frac{p^4}{4}\right\rangle+\langle V(x)^2\rangle+\ \text{Re}\ \langle p^2V(x)\rangle. \tag{13}$$

The calculation of the first two terms is straightforward while the calculation of the third term needs some elaboration:

$$\langle p^2V(x)\rangle=\langle\psi|p^2V(x)|\psi\rangle=\int_{-\infty}^{+\infty}dp\ \ p^2\psi(p)^*r(p), \tag{14}$$

where $\psi(p)^*$ is the conjugate of the wave function in the momentum representation and $r(p)=(1/\sqrt{2\pi K})\int_{-\infty}^{+\infty}dx\psi(x)V(x)e^{-ipx/K}$ is just the Fourier transform of the combination of $\psi(x)V(x)$. The statistics at times $\tau=(2\pi/\Omega)s$ will be plotted, with s the strobe number.

We choose the parameters as follows: $\lambda=0.4$, $\Omega=2.52$, $K=1$, and $\kappa=1000$. Then the conditions for localization and delocalization are $\varepsilon_m>0.016$ and $\varepsilon_m>0.315$, respectively.

The initial state is chosen to be a minimum uncertainty state, which has the wave function

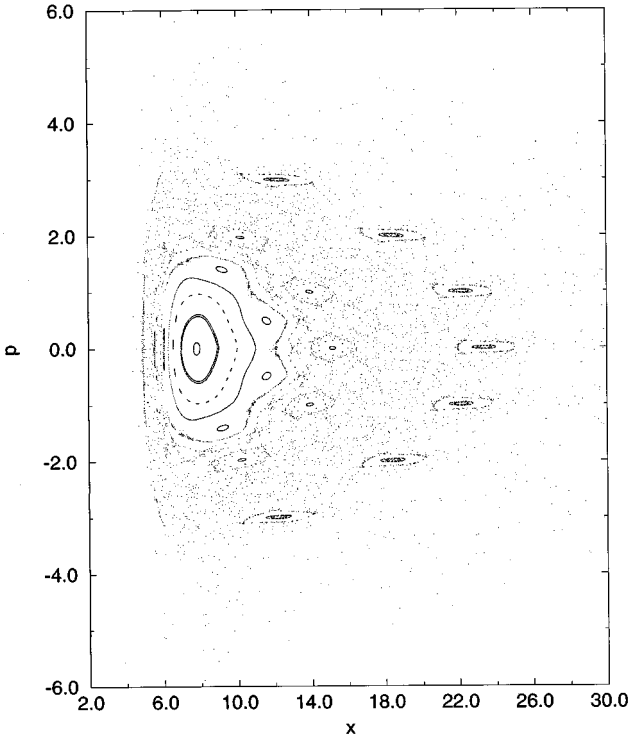

FIG. 1. Stroboscopic portrait of the system with $\kappa=1000$, $\lambda=0.4$, $K=1$, $\Omega=2.52$, $\varepsilon_m=0.1$, $p_0=0$, and $x_0=5$ (going up to 25 at an increment of 1).

$$\phi(x,0)=(2\pi\sigma_x)^{-1/4}\exp\left(i\frac{p_0x}{K}-\frac{(x-x_0)^2}{4\sigma_x}\right) \qquad (15)$$

with initial mean position x_0, initial mean momentum p_0, position variance σ_x, and momentum variance $\sigma_p=K^2/4\sigma_x$. In all the calculations, scaled dimensionless parameters are used.

Figure 1 is the stroboscopic portrait of the system with $\varepsilon_m=0.1$, $p_0=0$, and $x_0=5$ going up to 25 with an increment of 1. Obviously, if we choose x_0 to be bigger than 16, the motion will be in the globally chaotic regime.

Figure 2 is the classical and quantum distributions of the mean of the energy squared against the strobe number s under the conditions $\varepsilon_m=0.1, x_0=18, p_0=0, \sigma_x=1, \sigma_p=0.25$, with the dashed line representing the classical case and the solid line the quantum case. While the classical distribution diffuses indefinitely with time, the quantum distribution saturates quickly. Dynamical localization is clearly demonstrated.

Figure 3 is the classical and quantum distributions of the mean of the energy squared against the strobe number s under the same conditions as those in Fig. 2, but with $\varepsilon_m=0.35$ located in the delocalization regime. The dashed and solid lines represent the classical and quantum results, respectively. As the modulation exceeds both the classical chaos limit and the quantum border, the quantum distribution shows unbounded diffusion, and delocalization occurs.

Now let us have a look at the practical parameters corresponding to our simulation: for a Cesium atom $m=2.21\times10^{-25}$ kg, let $\alpha=4.9\times10^6$ m^{-1}, then $\omega_s=1.1\times10^4$ Hz, the modulation frequency is $\omega_m=2.75\times10^4$ Hz, while the initial position of the atoms is about 3.67 μm and

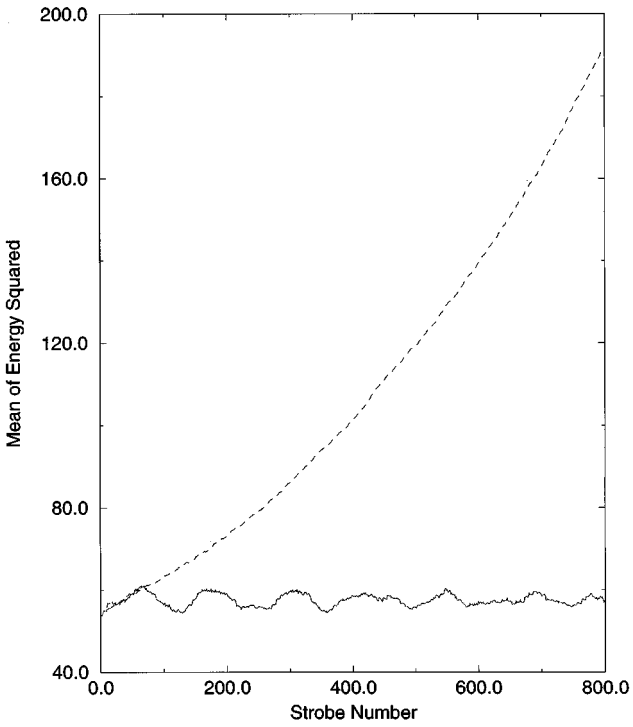

FIG. 2. Classical and quantum distributions of the mean of the energy squared against the strobe number s for the periodically modulated evanescent wave with $\kappa=1000$, $\lambda=0.4$, $K=1$, $\Omega=2.52$, $\varepsilon_m=0.1$, $x_0=18$, $p_0=0$, $\sigma_x=1$, and $\sigma_p=0.25$, with the dashed line representing the classical case and the solid line the quantum case.

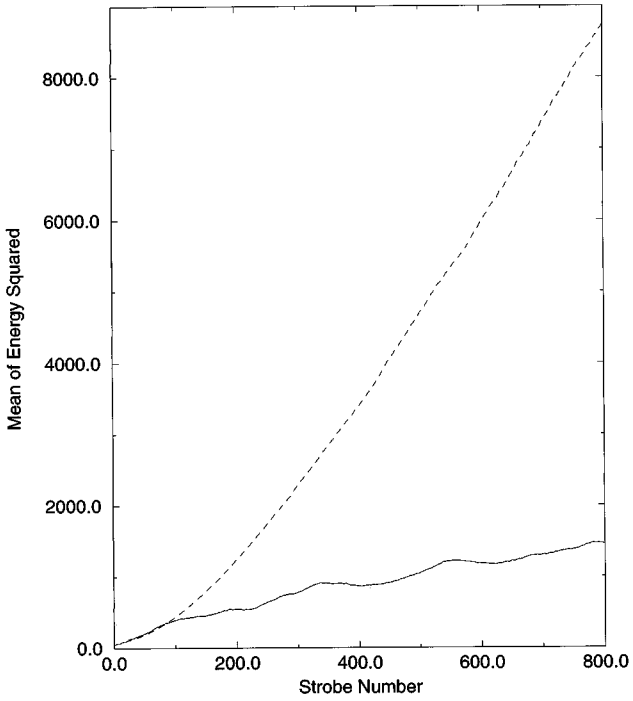

FIG. 3. Classical and quantum distributions of the mean of the energy squared against the strobe number s for the periodically modulated evanescent wave with $\kappa=1000$, $\lambda=0.4$, $K=1$, $\Omega=2.52$, $\varepsilon_m=0.35$, $x_0=18$, $p_0=0$, $\sigma_x=1$, and $\sigma_p=0.25$, with the dashed line representing the classical case and the solid line the quantum case.

the time at which the localization occurs is about 13.6 ms. The time scale is not too big, but we have to find a way to start the atoms very close to the surface and minimize the dissipation effects to allow the atoms enough time to show the above interesting phenomena.

But how can we measure the energy of the atom? According to Liston [24], if the evanescent wave forming the atomic mirror is coupled to an optical cavity, as the atom falls repeatedly onto the evanescent wave and is reflected away, it alters the phase of the light field. The magnitude of the phase change depends on the energy of the atom, the larger the energy, the smaller the phase change. Therefore if we allow the atom to bounce for some time period, the natural free evolution will result in an entangled state, where the parts of the atomic wave function with larger energies will be coupled to the parts of the field with smaller phase changes. Then the measurement of the phase quadrature of the light field will reveal information concerning the energy of the atom. An alternative way to see delocalization would be to wait until the atoms become sufficiently excited to penetrate the evanescent potential and adhere to the surface. This is analogous to ionization in chaotic atomic models. Localization would then correspond to a suppression of adhesion.

In summary, we have shown both localization and delocalization in the atomic gravitational cavity. The predicted effects rest entirely on coherence, so it is very important to keep the noise levels very low in order to observe them experimentally. Firstly, we have to suppress spontaneous emission by means of sufficiently large detuning. Secondly, we have to choose a good combination of parameters so that the surface adsorption [25] is small enough to allow long-time evolution of atoms. Finally, we have to keep other noise sources, such as intensity noise of the evanescent wave, very low as well.

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